No. Date. July 2, 2026

when adding up committees, there's a simple rule of thumbs that talents make a difference and follies make a sum.

Piet Hein

$$y = \frac{\ln(x)}{x}$$

1° Domain

$x>0$ because $\ln(x)$ has domain $x>0$ i $\in(0,\infty)$

2° Intercepts

y-int doesn't exist since $x>0$

x-int $\quad 0 = \frac{\ln(x)}{(x)} \iff \ln(x) = 0$

$\iff x = 1$

3° Vertical Asymptotes

Continuous where defined but the definition ends at 0

$$\lim_{x\to 0^+} \frac{\ln(x) \to -\infty}{x \to 0^+} = -\infty$$

we have a vertical asymptote at $x=0$ (approaching $-\infty$)

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4° Horizontal Asymptotes.

nothing in the negative direction because it is undefined ~~at~~ for $x \leq 0$

$$\lim_{x\to\infty} \frac{\ln(x) \to \infty}{x \to \infty} = \text{Lhôpital's rule} \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln(x))}{\frac{d}{dx}(x)} \Rightarrow$$

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0^+$$

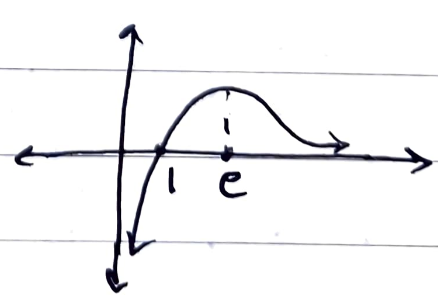
5° min/max/inc/dec

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln(x)}{x}\right) = \frac{\left[\frac{d}{dx}\ln(x)\right]x - \ln(x)\left[\frac{d}{dx}x\right]}{x^2} \Rightarrow$$

$$\frac{\frac{1}{x}\cdot x - \ln(x)\cdot 1}{x^2} \Rightarrow \frac{1-\ln(x)}{x^2}$$

$$\frac{dy}{dx} = \frac{1-\ln(x)}{x^2} \quad \begin{cases} x>0 & x^2 & & x<e \\ x=0 & 1-\ln(x) & \ln(x)=1 & x=e \\ x<0 & & & x>e \end{cases}$$

x	$(0,e)$	e	(e,∞)
$\frac{dy}{dx}$	$+$	0	$-$
y	$\uparrow$	max	$\downarrow$



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6° Concavity & inflection

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1-\ln(x)}{x^2}\right) \Rightarrow$$

$$\frac{\left[\frac{d}{dx}(1-\ln(x))\right]x^2 - (1-\ln(x))\left[\frac{d}{dx}x^2\right]}{x^4} \Rightarrow$$

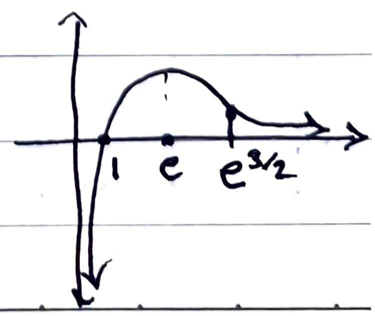
$$\frac{\left(-\frac{1}{x}\right)x^2 - (1-\ln(x))(2x)}{x^4} = \frac{-x-(1-\ln(x))2x}{x^4} \Rightarrow$$

$$\frac{-1-2+2\ln(x)}{x^3} = \frac{2\ln(x)-3}{x^3}$$

$$\begin{cases} x>0 \\ x=0 \\ x<0 \end{cases} \iff 2\ln(x)-3 \begin{matrix} >0 \\ =0 \\ <0 \end{matrix} \iff 2\ln(x) \begin{matrix} >3 \\ =3 \\ <3 \end{matrix}$$

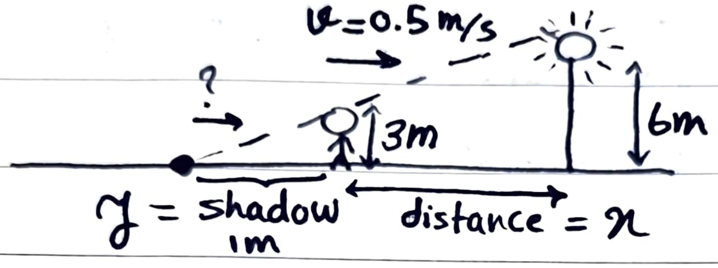
$$\iff \ln(x) \begin{matrix} > \frac{3}{2} \\ = \frac{3}{2} \\ < \frac{3}{2} \end{matrix} \iff \begin{matrix} x > e^{3/2} \\ x = e^{\frac{3}{2}} \\ x < e^{3/2} \end{matrix}$$

x	$(0, e^{3/2})$	$e^{3/2}$	$(e^{3/2}, \infty)$
$\frac{d^2y}{dx^2}$	$-$	0	$+$
y	$\cap$	Point of inflection	$\cup$



Related Rates :

Derivatives are rates of change, so this should really be called "related ~~rates~~ derivatives".



* pic not to scale!!!

How is the tip of the shadow moving when the shadow is 1m long?

→ At the instant that $y=1$, we have $\frac{dx}{dt} = -0.5$

→ We want to know $\frac{d}{dt}(x+y)$ at this moment.

How are x & y related?

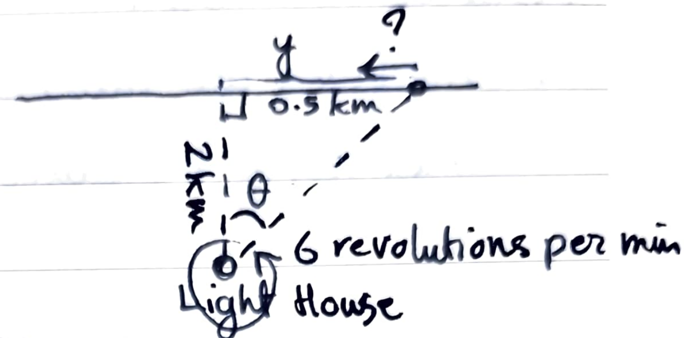
$$\frac{3}{y} = \frac{6}{x+y} \Rightarrow 6y = 3x + 3y \Rightarrow \boxed{x = 3}$$

$$\frac{dy}{dt} = \frac{dx}{dt} = -0.5 \quad \text{so:}$$

$$\frac{d}{dt}(x+y) = -0.5 - 0.5 = \boxed{-1 \text{ m/s}}$$

* A problem similar to this will be on the final exam.

Another example:



How is the beam moving along the shore when it is half a km from the nearest point on the shore to the lighthouse?

$$\frac{dy}{dt}\Big|_{y=0.5}$$

$$\tan\theta = \frac{y}{2}$$

$$6\text{rpm} \Rightarrow \frac{d\theta}{dt} = 2(6\pi) \quad 12\pi \ \text{radians}/\text{min}$$

when $y = 0.5$, $\tan(\theta) = \frac{1}{4}$

$$\Rightarrow \theta = \arctan\left(\frac{1}{4}\right) \Rightarrow$$

$$y = 2\tan\theta \Rightarrow$$

$$\frac{dy}{dt} = \left(\frac{d}{d\theta} 2\tan\theta\right)\left(\frac{d\theta}{dt}\right) \Rightarrow$$

$$2\sec^2(\theta) \cdot \frac{d\theta}{dt} \Rightarrow$$

$$2\sec^2(\theta) \cdot 12\pi \Rightarrow 24\pi \sec^2(\theta)$$

Therefore:

$$\frac{dy}{dt}\Big|_{y=0.5} = 24\pi\sec^2(\theta) = 24\pi(1+\tan^2(\theta)) \Rightarrow$$

[for θ at $y=0.5$]

$$24\pi\left(1+\left(\frac{1}{4}\right)^2\right) \Rightarrow$$

$$\boxed{\frac{24\pi \cdot 17}{16} \ \text{km}/\text{min}}$$

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Another example:

$$\frac{dr}{dt} = 1 \ m/s$$

How is the area enclosed by the ripple change after 3s?

$$A(r) = \frac{\pi r^2}{2} \quad \text{so} \quad \frac{dA}{dt} = \frac{d}{dt}\left(\frac{\pi r^2}{2}\right) = \left(\frac{d}{dr}\frac{\pi r^2}{2}\right) \cdot \frac{dr}{dt}$$

& we know $r\big|_{t=3} = 3(1) = 3 \ m$

$$\text{so} \Rightarrow \frac{2\pi r}{2} \cdot \frac{dr}{dt}$$

$$\left.\frac{dA}{dt}\right|_{t=3} = \cancel{4}\pi(3)(1) = 3\pi \ m^2/s$$

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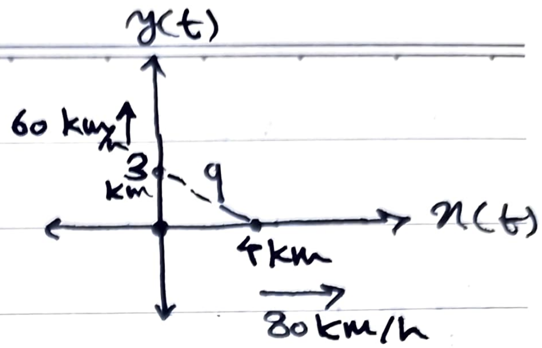


Another example:

one car starts at $y=3$ & moves up the y-axis at a speed of 60 km/h,

Another car starts at $x=4$ & moves at 80 km/h on the x-axis.

How is the distance between them changing after $\frac{1}{2}$ hr?

$$y(t) = 3 + 60t$$

$$x(t) = 4 + 80t$$

$$q(t) = \sqrt{x^2+y^2} = \sqrt{(3+60t)^2+(4+80t)^2}$$

$$25 + 10^3 t + 10^4 t^2$$

$$\left.\frac{dq}{dt}\right|_{t=\frac{1}{2}} = \frac{d}{dt}\left(\sqrt{x(t)^2+y(t)^2}\right) = \frac{d}{dt}\left(\sqrt{25+10^3t+10^4t^2}\right) \Rightarrow$$

$$\frac{d}{dt}(25+10^3t+10^4t^2)^{\frac{1}{2}} =$$

$$\frac{1}{2}(25+10^3t+10^4t^2)^{-1/2}\cdot(10^3+2\times10^4t) \Rightarrow$$

$$\frac{1}{2}\cdot\frac{1}{\sqrt{3025}}\cdot 11\times10^3 \Rightarrow \frac{5500}{\sqrt{3025}} = \boxed{100 \text{ km/hr}}$$

Another example :

A spherical balloon is being inflated at a rate of $2\ m^2/s$. How is the surface area of the balloon changing at the instant that its volume is $\frac{32}{3}\pi\ m^3/s$:

$$\frac{dSA}{dt} = ?$$

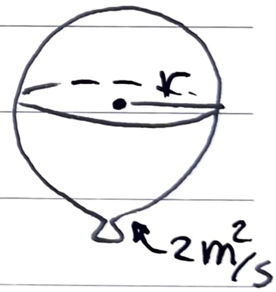
$SA = 4\pi r^2$ $\quad V = \frac{4}{3}\pi r^3$ $\quad \frac{dv}{dt} = 2\ m^3/s$

→ Filter information about V through r to get information about SA.

when $\frac{32}{3}\pi\ m^3$, r is : $\frac{4}{3}\pi r^3 = \frac{32}{3}\pi \Rightarrow r^3 = \frac{32}{4} = 8$

$\boxed{r = 2m}$

when $\frac{dv}{dt} = 2 m^2/s$, $\frac{dr}{dt} = ? \Rightarrow$



$$\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt} = \frac{4}{3}\pi 3r^2 = 4\pi r^2 \frac{dr}{dt}$$

when $r = 2$:

$$2 = \frac{dv}{dt}\bigg|_{r=2} = \pi r^2 \frac{dr}{dt}\bigg|_{r=2} = 4\pi \cdot \frac{dr}{dt} \Rightarrow$$

$$\frac{dr}{dt}\bigg|_{r=2} = \frac{2}{4\pi} = \boxed{\frac{1}{2\pi}}$$

$$\frac{dSA}{dt}\bigg|_{V=\frac{32}{3}\pi} = \frac{dSA}{dt}\bigg|_{r=2} = \frac{d}{dt}(4\pi r^2)\bigg|_{r=2} =$$

$$\left[\frac{d}{dr}(4\pi r^2)\right] \cdot \frac{dr}{dt} = 8\pi r \cdot \frac{dr}{dt}\bigg|_{r=2} = \boxed{8\ m^2/s}$$